New Results on Periodic Solutions of Delayed Nicholson's Blowflies Models

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Abstract: This paper is concerned with a class of Nicholson's blowflies models with a nonlinear density-dependent mortality term. We use coincidence degree theory and give several sufficient conditions which guarantee the existence of positive periodic solutions of the model. Moreover, we give an example to illustrate our main results.

Keywords: Nicholson's blowflies Model; positive periodic solution; coincidence degree; nonlinear density-dependent mortality term.

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1. Introduction

In biological applications, Gurney et al. [1] introduced a mathematical model

$$N'(t) = -\delta N(t) + pN(t-\tau)e^{-aN(t-\tau)}, \tag{1.1}$$

to describe the population of the Australian sheep-blowfly and to agree with the experimental data obtained in [2]. Here, $N(t)$ is the size of the population at time t, p is the maximum per capita daily egg production, $\frac{1}{a}$ is the size at which the population reproduces at its maximum rate, δ is the per capita daily adult death rate, and τ is the generation time. The model and

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its modifications have also been later used to describe population growth of other species (see, e.g., Cooke et al. [3]), and thus, have been extensively and intensively studied (see, e.g., [3-6]). In particular, there have been extensive results on the problem of the existence of positive periodic solutions for Nicholson's blowflies equation in the literature. We refer the reader to [7−8] and the references cited therein. In [7], Chen obtained the result of existence of periodic solutions of Nicholson's blowflies model of the form

$$N'(t) = -\delta(t)N(t) + P(t)N(t - \sigma(t))e^{-a(t)N(t-\tau(t))}, \tag{1.2}$$

where $\delta \in C(R, R), P, \sigma, \tau \in C(R, (0, +\infty))$ and $a \in C(R, (0, +\infty))$ are T-periodic functions with $\int_0^T \delta(t)dt > 0$. In [8], Li and Du researched the following generalized Nicholson's blowflies model:

$$N'(t) = -\delta(t)N(t) + \sum_{i=1}^{m} p_i(t)N(t - \tau_i(t))e^{-q_i(t)N(t-\tau_i(t))}, \tag{1.3}$$

where $\delta, p_i, q_i \in C(R^+, (0, +\infty))$ and $\tau_i \in C(R^+, R^+)$ are T-periodic functions for $i = 1, 2, \ldots, m$ with $\int_0^T \delta(t)dt > 0$. They established a sufficient and necessary condition for the existence of positive periodic solutions for (1.3).

Recently, as pointed out in L. Berezansky et al. [9], a new study indicates that a linear model of density-dependent mortality will be most accurate for populations at low densities, and marine ecologists are currently in the process of constructing new fishery models with nonlinear density-dependent mortality rates. Therefore, L. Berezansky et al. [9] proposed an open problems: Reveal the dynamic behaviors of the Nicholson's blowflies model with a nonlinear density-dependent mortality term as follows:

$$N'(t) = -D(N) + PN(t - \tau)e^{-N(t-\tau)}, \tag{1.4}$$

where P is a positive constant and function D might have one of the following forms: $D(N) = aN/(N + b)$ or $D(N) = a - be^{-N}$. Furthermore, B. Liu [10] obtain permanence for models (1.4) with $D(N) = aN/(N + b)$, and W. Wang [11] studied the existence of positive periodic solutions for the models (1.4) with $D(N) = a - be^{-N}$. However, to the best of our knowledge, few authors have considered the problem for positive periodic solutions of Nicholson's blowflies models (1.4) with $D(N) = aN/(N + b)$. Thus, it is worthwhile to continue to investigate the existence of positive periodic solutions of (1.4) in this case.

The main purpose of this paper is to give the conditions for the existence of the positive periodic solutions for Nicholson's blowflies models (1.4) with $D(N) = aN/(N + b)$. Since the coefficients and delays in differential equations of population and ecology problems are usually time-varying in the real world, so we'll consider the delayed Nicholson's blowflies models with a nonlinear density-dependent mortality term:

$$N'(t) = -\frac{a(t)N(t)}{b(t) + N(t)} + c(t)N(t - \tau(t))e^{-\gamma(t)N(t-\tau(t))}, \tag{1.5}$$

where $a, b, c, \gamma, \tau \in C(R, (0, \infty))$ are positive T-periodic functions. It is obvious that when $D(N) = aN/(N + b)$, (1.4) is a special case of (1.5).

Throughout this paper, given a bounded continuous function g defined on R, let g^+ and g^- be defined as

$$g^- = \inf_{t\in R} g(t), \quad g^+ = \sup_{t\in R} g(t).$$

The remaining part of this paper is organized as follows. In section 2, we shall derive new sufficient conditions for checking the existence of the positive periodic solutions of model (1.5). In Section 3, we shall give an example and a remark to illustrate our results obtained in the previous sections.

2. Existence of Positive Periodic Solutions

For convenience, we will let $X = Z = \{x \in C(R, R) : x(t + T) = x(t) \text{ for all } t \in R\}$ be Banach spaces equipped with the norm $||\cdot||$, where $||x|| = \max_{t\in[0,T]} |x(t)|$. For any $x \in X$, we denote

$$\Delta(x, t) = -\frac{a(t)}{b(t) + e^{x(t)}} + c(t)e^{x(t-\tau(t))-x(t)-\gamma(t)e^{x(t-\tau(t))}}.$$

Because of periodicity, it is easy to see that $\Delta(x, \cdot) \in C(R, R)$ is T-periodic. Let

$$L : D(L) = \{x \in X : x \in C^1(R, R)\} \ni x \longmapsto x' \in Z,$$

$$P : X \ni x \longmapsto \frac{1}{T}\int_0^T x(s)ds \in X,$$

$$Q : Z \ni z \longmapsto \frac{1}{T}\int_0^T z(s)ds \in Z,$$
$$N : X \ni x \longmapsto \Delta(x, \cdot) \in Z.$$

From the definitions of the above operators. It is easy to see that

$$ImL = \{x | x \in Z, \int_0^T x(s)ds = 0\}, KerL = R, ImP = KerL \text{ and } KerQ = ImL.$$

Thus, the operator L is a Fredholm operator with index zero.

In order to study the existence of positive periodic solutions, we first introduce the Continuation Theorem as follows:

Lemma 1 (Continuation Theorem) [12]. Let X and Z be two Banach spaces. Suppose that $L : D(L) \subset X \to Z$ is a Fredholm operator with index zero and $N : X \to Z$ is L -compact on $\overline{\Omega}$, where Ω is an open subset of X. Moreover, assume that all the following conditions are satisfied:

(1) $Lx \neq \lambda Nx$, for all $x \in \partial\Omega \cap D(L), \lambda \in (0, 1)$;

(2) $Nx \notin ImL$, for all $x \in \partial\Omega \cap KerL$;

(3) The Brouwer degree

$$\deg\{QN, \Omega \cap KerL, 0\} \neq 0.$$

Then equation $Lx = Nx$ has at least one solution in $domL \cap \overline{\Omega}$.

Our main result is given in the following theorem.

Theorem 1. Set

$$A = 2\int_0^T \frac{a(t)}{b(t)}dt, \quad B = \int_0^T c(t)dt, \ \ln\frac{2B}{A} > A, \text{ and } 1 > \frac{c^+}{a^-\gamma^-e}. \tag{2.1}$$

Then (1.5) has a positive T-periodic solution.

Proof. Set $N(t) = e^{x(t)}$, then (1.5) can be rewritten as

$$\begin{aligned} x'(t) &= -\tfrac{a(t)}{b(t)+e^{x(t)}} + c(t)e^{x(t-\tau(t)) - x(t) - \gamma(t)e^{x(t-\tau(t))}} \\ &= \Delta(x, t). \end{aligned} \tag{2.2}$$

Then, to prove Theorem 1, it suffices to show that equation (2.2) has at least one T-periodic solution. Denoting by $L_P^{-1} : ImL \to D(L) \cap KerP$ the inverse of $L|_{D(L)\cap KerP}$, we have

$$L_P^{-1}y(t) = -\frac{1}{T}\int_0^T \int_0^t y(s)dsdt + \int_0^t y(s)ds. \tag{2.3}$$

To apply Lemma 1, we first claim that N is L-compact on $\overline{\Omega}$, where Ω is a bounded open subset of X. From (2.3), it follows that

$$QNx = \frac{1}{T}\int_0^T Nx(t)dt = \frac{1}{T}\int_0^T [-\frac{a(t)}{b(t)+e^{x(t)}} + c(t)e^{x(t-\tau(t))-x(t)-\gamma(t)e^{x(t-\tau(t))}}]dt, \tag{2.4}$$

$$\begin{aligned} L_P^{-1}(I-Q)Nx = & \int_0^t Nx(s)ds - \frac{t}{T}\int_0^T Nx(s)ds - \frac{1}{T}\int_0^T\int_0^t Nx(s)dsdt \\ & +\frac{1}{T}\int_0^T\int_0^t QNx(s)dsdt. \end{aligned} \tag{2.5}$$

Obviously, QN and $L_P^{-1}(I-Q)N$ are continuous. It is not difficult to show that $L_P^{-1}(I-Q)N(\overline{\Omega})$ is compact for any open bounded set $\Omega \subset X$ by using the Arzela-Ascoli theorem. Moreover, $QN(\overline{\Omega})$ is clearly bounded. Thus, N is L-compact on $\overline{\Omega}$ with any open bounded set $\Omega \subset X$.

Considering the operator equation $Lx = \lambda Nx, \lambda \in (0,1)$, we have

$$x'(t) = \lambda\Delta(x,t). \tag{2.6}$$

Assume that $x \in X$ is a solution of (2.6) for some $\lambda \in (0,1)$. Then

$$\begin{aligned} \int_0^T |c(t)e^{x(t-\tau(t))-x(t)-\gamma(t)e^{x(t-\tau(t))}}|dt &= \int_0^T c(t)e^{x(t-\tau(t))-x(t)-\gamma(t)e^{x(t-\tau(t))}}dt \\ &= \int_0^T \frac{a(t)}{b(t)+e^{x(t)}}dt \\ &= \int_0^T |\frac{a(t)}{b(t)+e^{x(t)}}|dt \\ &< \int_0^T \frac{a(t)}{b(t)}dt. \end{aligned} \tag{2.7}$$

It follows from (2.6) and (2.7) that

$$\begin{aligned} \int_0^T |x'(t)|dt &\le \lambda\int_0^T |c(t)e^{x(t-\tau(t))-x(t)-\gamma(t)e^{x(t-\tau(t))}}|dt + \lambda\int_0^T |\frac{a(t)}{b(t)+e^{x(t)}}|dt \\ &< 2\int_0^T \frac{a(t)}{b(t)}dt = A. \end{aligned} \tag{2.8}$$

Since $x \in X$, there exist $\xi, \eta \in [0,T]$ such that

$$x(\xi) = \min_{t\in[0,T]} x(t), \; x(\eta) = \max_{t\in[0,T]} x(t), \quad \text{and} \quad x'(\xi) = x'(\eta) = 0. \tag{2.9}$$

It follows from (2.7) and (2.8) that

$$\begin{aligned}
\frac{A}{2} &= \int_0^T \frac{a(t)}{b(t)}dt \\
&> \int_0^T \frac{a(t)}{b(t)+e^{x(t)}}dt \\
&= \int_0^T c(t)e^{x(t-\tau(t))-x(t)-\gamma(t)e^{x(t-\tau(t))}}dt \\
&\geq e^{x(\xi)-x(\eta)-\gamma^+e^{x(\eta)}}\int_0^T c(t)dt \\
&= Be^{x(\xi)-x(\eta)-\gamma^+e^{x(\eta)}},
\end{aligned}$$

which implies that

$$x(\xi) < \ln\frac{A}{2B} + x(\eta) + \gamma^+e^{x(\eta)}.$$

Using (2.8) yields

$$x(t) \leq x(\xi) + \int_0^T |x'(t)|dt < \ln\frac{A}{2B} + x(\eta) + \gamma^+e^{x(\eta)} + A.$$

In particular,

$$x(\eta) < x(\xi) + \int_0^T |x'(t)|dt < \ln\frac{A}{2B} + x(\eta) + \gamma^+e^{x(\eta)} + A.$$

It follows that

$$x(\eta) > \ln(\frac{1}{\gamma^+}(\ln\frac{2B}{A} - A)).$$

Again from (2.8), we have

$$x(t) \geq x(\eta) - \int_0^T |x'(t)|dt > \ln(\frac{1}{\gamma^+}(\ln\frac{2B}{A} - A)) - A := H_1. \tag{2.10}$$

Since $x'(\xi) = 0$, from (2.7), we obtain

$$\frac{a(\xi)}{b(\xi)+e^{x(\xi)}} = c(\xi)e^{x(\xi-\tau(\xi))-x(\xi)-\gamma(\xi)e^{x(\xi-\tau(\xi))}}. \tag{2.11}$$

Hence, from (2.11) and the fact that $\sup\limits_{u\geq 0} ue^{-u} = \frac{1}{e}$, we have

$$\frac{e^{x(\xi)}}{b^+ + e^{x(\xi)}} \leq \frac{e^{x(\xi)}}{b(\xi)+e^{x(\xi)}} = \frac{c(\xi)}{a(\xi)\gamma(\xi)}\gamma(\xi)e^{x(\xi-\tau(\xi))}e^{-\gamma(\xi)e^{x(\xi-\tau(\xi))}} \leq \frac{c^+}{a^-\gamma^-e}. \tag{2.12}$$

Noting that $\frac{u}{b^+ + u}$ is strictly monotone increasing on $[0, +\infty)$ and

$$\sup_{u\geq 0} \frac{u}{b^+ + u} = 1 > \frac{c^+}{a^-\gamma^- e},$$

it is clear that there exists a constant $k > 0$ such that

$$\frac{u}{b^+ + u} > \frac{c^+}{a^-\gamma^- e} \quad \text{for all} \quad u \in [k, +\infty). \tag{2.13}$$

In view of (2.12) and (2.13), we get

$$e^{x(\xi)} \leq k \quad \text{and} \quad x(\xi) \leq \ln k. \tag{2.14}$$

Then, we can choose a sufficiently large positive constant $H_2 > \ln k$ such that

$$x(t) < H_2 \quad \text{and} \quad \ln b^+ < H_2. \tag{2.15}$$

Let $H > \max\{|H_1|, H_2\}$ be a fix constant such that

$$e^H > \frac{1}{\gamma^-}(H + \ln \frac{2B}{C}) \quad \text{with } C = \int_0^T a(t)dt,$$

and define $\Omega = \{x \in X : ||x|| < H\}$. Then (2.10) and (2.15) imply that there is no $\lambda \in (0, 1)$ and $x \in \partial\Omega$ such that $Lx = \lambda Nx$.

When $x \in \partial\Omega \cap KerL = \partial\Omega \cap R$, $x = \pm H$. Then

$$QN(-H) > 0 \quad \text{and} \quad QN(H) < 0. \tag{2.16}$$

Otherwise, if $QN(-H) \leq 0$, it follows from (2.4) that

$$\begin{aligned}
\frac{A}{2} &= \int_0^T \frac{a(t)}{b(t)}dt \\
&> \int_0^T \frac{a(t)}{b(t) + e^{-H}}dt \\
&\geq \int_0^T c(t)e^{-\gamma(t)e^{-H}}dt \\
&\geq e^{-\gamma^+ e^{-H}} \int_0^T c(t)dt \\
&= Be^{-\gamma^+ e^{-H}},
\end{aligned}$$

which implies

$$-H \geq \ln(\frac{1}{\gamma^+}\ln\frac{2B}{A}) > \ln(\frac{1}{\gamma^+}(\ln\frac{2B}{A} - A)) - A = H_1.$$

This is a contradiction and implies that $QN(-H) > 0$.

If $QN(H) \geq 0$, it follows from (2.4) that

$$\begin{aligned}\frac{C}{2}e^{-H} &= \int_0^T \frac{a(t)}{2e^H}dt \\ &< \int_0^T \frac{a(t)}{b(t)+e^H}dt \\ &\leq \int_0^T c(t)e^{-\gamma(t)e^H}dt \\ &\leq e^{-\gamma^- e^H}\int_0^T c(t)dt \\ &= Be^{-\gamma^- e^H}.\end{aligned}$$

Consequently,

$$e^H < \frac{1}{\gamma^-}(H + \ln\frac{2B}{C}),$$

a contradiction to the choice of H. Thus, $QN(H) < 0$.

Furthermore, define continuous function $H(x,\mu)$ by setting

$$H(x,\mu) = -(1-\mu)x + \mu\frac{1}{T}\int_0^T[-\frac{a(t)}{b(t)+e^x} + c(t)e^{-\gamma(t)x}]dt.$$

It follows from (2.16) that $xH(x,\mu) \neq 0$ for all $x \in \partial\Omega \cap kerL$. Hence, using the homotopy invariance theorem, we obtain

$$\begin{aligned}\deg\{QN, \Omega\cap kerL, 0\} &= \deg\{\tfrac{1}{T}\textstyle\int_0^T[-\frac{a(t)}{b(t)+e^x} + c(t)e^{-\gamma(t)x}]dt, \Omega\cap kerL, 0\} \\ &= \deg\{-x, \Omega\cap kerL, 0\} \neq 0.\end{aligned}$$

In view of all the discussions above, we conclude from Lemma 1 that Theorem 1 is proved.

3. An Example

In this section we present an example to illustrate our results.

Example 3.1. Consider the delayed periodic Nicholson's blowflies models with a nonlinear density-dependent mortality term:

$$N'(t) = -\frac{(2+\sin t)N(t)}{2+\sin t + N(t)} + (\frac{e^{4\pi}}{4}+1)(4+\cos t)N(t-e^{4\pi+\sin t})e^{-e^{4\pi+|\sin t|}N(t-e^{4\pi+\sin t})} \quad (3.1)$$

has a positive 2π-periodic solution.

Proof. By (3.1), we have

$$a(t) = b(t) = 2 + \sin t, c(t) = (\frac{e^{4\pi}}{4} + 1)(4 + \cos t), \gamma(t) = e^{4\pi + |\sin t|},$$

then

$$A = 2\int_0^{2\pi} \frac{a(t)}{b(t)} dt = 4\pi, B = \int_0^{2\pi} c(t)dt = 8\pi + 2\pi e^{4\pi},$$

$$a^- = 1, c^+ = 5(\frac{e^{4\pi}}{4} + 1), \gamma^- = e^{4\pi}.$$

Clearly,

$$\ln \frac{2B}{A} = \ln(e^{4\pi} + 4) > 4\pi = A, \frac{c^+}{a^- \gamma^- e} = \frac{5}{4e} + \frac{5}{e^{4\pi+1}} < 1,$$

it means that conditions in Theorem 1 hold. Hence, the equation (3.1) has a positive 2π-periodic solution.

Remark 3.1. (3.1) is a kind of delayed periodic Nicholson's blowflies models with a nonlinear density-dependent mortality term, but as far as we know there are not results can be applicable to (3.1) to obtain the existence of 2π-periodic solutions. This implies that the results of this paper are essentially new.

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